Non-existence of unicellular operators on certain non-separable Banach spaces

By J. SZŰCS, E. MÁTÉ in Szeged, and C. FOIAŞ in Bucharest[1])

1. Professor B. SZ.-NAGY has raised the following problem: Does there exist a unicellular operator[2]) on a non-separable (real or complex) Banach space B? The following theorem gives a partial answer to this question.

Theorem 1. *In the following cases there exists no unicellular operator on B:*

(a) $\dim B > \aleph_1$;

(b) B^* *is non-separable in its w^*-topology i.e. there exists no countable subset in B^* which is dense in the w^*-topology*; [3])

(c) B *is reflexive* [4]) *and non-separable.*

Proof. *Part* (a). Let ω_1 be the smallest non-countable ordinal number. For every $\beta < \omega_1$ we define by transfinite induction an element x_β of B in the following way: Choose $x_1 \in B$ arbitrarily and if x_β has already been defined for $\beta < \alpha$ set $M_\alpha = \bigvee_{\beta < \alpha} x_\beta$, [5]) M_α is a separable subspace of B. Since B is not separable we can choose an element $y \in B$ for which $d = \inf_{x \in M_\alpha} \|x - y\| > 0$. Setting $x_\alpha = d^{-1} y$ we have

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1) Theorem 1, its proof and the second proof of Theorem 2 is due to the first, Remark 1 to the second, and Theorem 2, its first proof with the lemma and Remark 2 to the third author.

2) In this paper "operator" always means "linear bounded operator". The operator T is said to be unicellular if for any two closed invariant subspaces for T, say M and N, we have either $M \subset N$ or $N \subset M$.

3) B^* is the space of all bounded linear functionals on B; the w^* ("weak star") topology of B^* is the topology of the pointwise convergence of elements as functions on B.

4) I.e. each element $F \in B^{**}$ is of form $F(f) = f(x)$ with varying $f \in B^*$ and fixed $x \in B$, the choice of this latter depending, of course, on F.

5) If X is a subset of B we denote by $\bigvee_{x \in X} x$ the (closed) subspace of B spanned by the elements $x \in X$. If $\mathfrak{X}$ is a set of subsets of B, we denote by $\bigvee_{X \in \mathfrak{X}} X$ the subspace of B spanned by the elements x of $\bigcup_{X \in \mathfrak{X}} X$.

obviously $\inf_{x \in M_\alpha} \|x - x_\alpha\| = 1$. Thus for $\alpha < \omega_1$, $\beta < \omega_1$, $\alpha \neq \beta$ we have $\|x_\alpha - x_\beta\| \geqq 1$, and hence the dimension of $M = \bigvee_{\alpha < \omega_1} x_\alpha$ is equal to $\aleph_1$.

Suppose T is a linear operator on B and set $N = \bigvee_{n=0}^{\infty} T^n M$. Then N is invariant for T and clearly $\dim N = \aleph_1$, and hence $N \neq B$. Choose $z \in B \setminus N$ and set $N' = \bigvee_{n=0}^{\infty} T^n z$. Then N' is invariant for T and N' is not contained in N. On the other hand, since $\dim N > \dim N'$, N is not contained in N'. Hence T is not unicellular.

Part (b). Let T be any operator on B, x a non-zero element of B, and set $M = \bigvee_{n=0}^{\infty} T^n x$. Consider a strongly dense sequence $\{x_n\}_{n=0}^{\infty}$ in M. Choose a sequence of elements f_n of B^* with $f_n(x_n) = \|x_n\|$, $\|f_n\| = 1$. Let y be an element of M for which $f_n(y) = 0$ $(n = 0, 1, \ldots)$. Since $\{x_n\}_{n=0}^{\infty}$ is dense in M we have $\liminf_{n \to \infty} \|y - x_n\| = 0$. So

$$\|y\| \leqq \liminf_{n\to\infty} \left(\|y - x_n\| + \|x_n\|\right) = \liminf_{n\to\infty} \left(\|y - x_n\| + f_n(x_n)\right) =$$

$$= \liminf_{n\to\infty} \left(\|y - x_n\| + f_n(x_n - y)\right) \leqq \liminf_{n\to\infty} 2\|y - x_n\| = 0,$$

and hence $y = 0$.

Let N denote the w^*-closed linear subspace spanned by the elements of form $x \to f_n(T^i x)$ $(i, n = 0, 1, \ldots)$ in B^*. The preceding result means that ${}^{\perp}N \cap M = \{0\}$. [6])

On the other hand, ${}^{\perp}N$ is not trivial since $N \neq B^*$ [4, p. 65]. ${}^{\perp}N$ is invariant for T, and obviously neither ${}^{\perp}N$ nor M is contained in the other. Hence T is not unicellular.

Part (c). First of all we observe that B^* is not separable in the norm topology [4, p. 65]. Hence by MAZUR's theorem [3] on the closedness in the w-topology, of a convex and strongly closed subset of B^*, B^* is not w-separable. Since B is reflexive, the w and the w^*-topologies coincide, so B^* is not w^*-separable and Part (c) follows from Part (b).

Remark 1. Part (c) can also be proved without using Part (b) and the Mazur theorem. Define T, x, M, x_n and f_n as in the proof of Part (b). Since by [2, p. 65] B^* is not separable in the norm topology, and since B is reflexive, there exists a non-zero element z of B for which $f_n(T^i z) = 0$ $(i, n = 0, 1, \ldots)$. Set $L = \bigvee_{i=0}^{\infty} T^i z$.

[6]) For any subset X of a Banach space B we define

$$X^{\perp} = \{f\colon\ f \in B^*,\ f(x) = 0 \quad \text{for all } x \in B\},$$

and for any subset F of B^* we define

$${}^{\perp}F = \{x\colon\ x \in B,\ f(x) = 0 \quad \text{for all } f \in F\}.$$

Then L is a non-trivial invariant subspace for T and by the proof of Part (b) we have $L \cap M = \{0\}$. Since M too is a non-trivial invariant subspace for T, T is not unicellular.

2. In this section we formulate a theorem similar to Theorem 1.

Theorem 2. *If C is a non-separable real or complex Banach space and S is a bounded linear operator on C then $T=S^*$ is not unicellular on $B=C^*$.*

We give first a direct proof of Theorem 2 and then we give a proof along the line of the proof for Part (b) of Theorem 1.

First proof. Let us suppose that S^* is unicellular. Then S is also unicellular (see the Lemma below). Now, for a unicellular operator, say S, every non-trivial invariant subspace X is separable. Indeed, take a $y \in C$, $y \notin X$, and set

$$Y = \bigvee_{j=0}^{\infty} S^j y.$$

Then Y is an invariant subspace for S, and by virtue of $Y \not\subset X$ we must have $X \subset Y$. As Y is separable so must be X.

If S has no non-trivial invariant subspace then $C = \bigvee_{j=0}^{\infty} T^j x$ for any $x \in B$, $x \neq 0$, and hence C is separable. Thus if C *is non separable* then there exists a non-trivial invariant subspace X for S; therefore X is separable. Then $(B/X)^*$ is isomorphic to $X^\perp$, which is invariant for T^* and is different from C^* (the assumption of the contrary implies $X=\{0\}$). But S^* is unicellular so that $X^\perp$ is separable. Therefore $(C/X)^*$ is separable, hence C/X too is separable [2, p. 65]. Now the fact that X and C/X are separable implies that C is separable: contradiction. This completes the proof.

Second proof. Consider C as a subspace of $C^{**}=B^*$, and choose a sequence of elements f_n of C with the same properties as in Part (b) except that here we require $\|f_n\| \to 1$ as $n \to \infty$ instead of $\|f_n\|=1$; this is enough to conclude $y=0$. Then $N \cap C$ is obviously the w^*-closed subspace of C spanned by the elements $S^i f_n$ $(i, n=0, 1, \ldots)$. The w^*-topology of C^{**} coincides on C with the w-topology of C, and so by Mazur's theorem [3] $N \cap C \neq C$, which implies $N \neq B^*$. The same argument as used at the end of the proof of Part (b) completes the proof.

In the first proof we used the following

Lemma. *If S^* is unicellular then so is S.*

Proof. Let $X, Y \subset C$ be two (linear closed) subspaces of C, invariant for S. Then $X^\perp$ and $Y^\perp$ are subspaces of C^*, invariant for S^*. Since S^* is unicellular we may assume inclusion in one way or the other; if we have e.g.

$$X^{\perp} \subset Y^{\perp} \quad \text{then} \quad Y = {}^{\perp}(Y^{\perp}) \subset {}^{\perp}(X^{\perp}) = X,$$

which was to be shown.

Remark 2. Obviously, in the lemma C may also be separable. However the assertion remains no more valid if S and S^* are interchanged. This can be illustrated by the example in which $C = L^1(0, 1)$, $C^* = L^\infty(0, 1)$ and

$$[Sf](x) = \int_0^x f(t)\,dt \quad (0 < x < 1,\ f \in L^1(0,1)).$$

Then S is unicellular since its invariant (linear and closed) subspaces are

$$C_\alpha = \{f\colon\ f \in L^1(0, 1) \text{ and } f(t) = 0 \text{ a.e. if } 0 < t < \alpha\},$$

where $0 \leqq \alpha \leqq 1$ (see [1]).

Consider the element $g \in L^\infty(0, 1)$ that is equal to 1 a.e. on $(0, 1)$ and form the invariant subspace $M = \bigvee_{n=0}^{\infty} S^{*n} g$ of S^*. Compare M to the subspace

$$C_{\frac{1}{2}}^{\perp} = \left\{ f\colon f \in L^\infty(0,1),\ f(x) = 0 \text{ a.e. if } \frac{1}{2} < x < 1 \right\},$$

which is invariant under S^*. Since M is separable we have $M \not\supset C_{\frac{1}{2}}^{\perp}$; on the other hand, it is obvious that $M \not\subset C_{\frac{1}{2}}^{\perp}$; therefore S^* is not unicellular.

We do not know wether or not Theorem 2 holds for every bounded linear operator T on C^*. Thus the following problem is open: *Does there exist a unicellular operator on the conjugate of a non-separable Banach space C?*

References

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(Received June 23 (part 1)*; November 30* (part 2)*, 1969)*